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Topologies on $\mathbb{R}^n$ induced by smooth subsets

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Abstract

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If $\mathcal{S}$ is a collection of subsets of $\mathbb{R}^n$, let $\mathcal{T}_{\mathcal{S}}$ denote the largest topology on $\mathbb{R}^n$ which restricts to the standard topology on each element of $\mathcal{S}$, and let $\mathcal{H}_{\mathcal{S}}$ denote the homeomorphism group of $\mathbb{R}^n$ with the topology $\mathcal{T}_{\mathcal{S}}$. Let $\mathcal{T}_{\text{std}}$ denote the standard topology on $\mathbb{R}^n$ and let $\mathcal{H}_{\text{std}}$ denote the homeomorphism group of $\mathbb{R}^n$ with the standard topology.

Theorem 1. *If $\mathcal{S}$ is any collection of subsets of $\mathbb{R}^n$ which contains all C^1 regular 1-manifolds, then $\mathcal{T}_{\mathcal{S}} = \mathcal{T}_{\text{std}}$.*

A natural collection of subsets of $\mathbb{R}^n$ called *smooth* sets is defined which includes the zero set of every nonconstant polynomial and every C^2 regular submanifold of $\mathbb{R}^n$ of dimension $<n$.

Theorem 2. *If $\mathcal{S}$ is the collection of all smooth subsets of $\mathbb{R}^n$, then $\mathcal{T}_{\mathcal{S}}$ is strictly larger than $\mathcal{T}_{\text{std}}$ and $\mathcal{H}_{\mathcal{S}}$ is strictly smaller than $\mathcal{H}_{\text{std}}$.*

Theorem 3. *There is an injective function $f:\mathbb{R}^n \to \mathbb{R}^n$ which is discontinuous at each point of a countable dense subset of $\mathbb{R}^n$, and whose restriction to each smooth subset of $\mathbb{R}^n$ is continuous.*

Keywords: $C^1$1-manifolds, smooth sets, induced topologies, induced homeomorphism groups, Taylor's formula.

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1. Introduction

At the 1988 Spring Topology Conference in Gainesville, FL, Otto Laback, a physicist from Graz Technical University in Graz, Austria, posed the following question. Given that we can directly observe only certain subsets of $\mathbb{R}^n$ (such as smoothly embedded 1-manifolds corresponding to particle paths), what possibly nonstandard topologies on $\mathbb{R}^n$ are compatible with the usual topology on physically observable subsets? He also wondered how the homeomorphism group of $\mathbb{R}^n$ with such a nonstandard topology compares to the standard homeomorphism group. The following definition allows us to give a precise formulation of a version of this question.

Let $\mathcal{S}$ be a collection of subsets of $\mathbb{R}^n$. Define $\mathcal{T}_{\mathcal{S}}$ to be $\{U \subset \mathbb{R}^n: U \cap S$ is a relatively open subset of S for each $S \in \mathcal{S}\}$. Then $\mathcal{T}_{\mathcal{S}}$ is the largest topology on $\mathbb{R}^n$ which restricts to the standard topology on each element of $\mathcal{S}$. Define $\mathcal{H}_{\mathcal{S}}$ to be the homeomorphism group of $\mathbb{R}^n$ with the topology $\mathcal{T}_{\mathcal{S}}$. Let $\mathcal{T}_{\text{std}}$ denote the standard topology on $\mathbb{R}^n$, and let $\mathcal{H}_{\text{std}}$ denote the homeomorphism group of $\mathbb{R}^n$ with the standard topology.

We now state a version of Laback's question.

Question. For which collections $\mathcal{S}$ of subsets of $\mathbb{R}^n$ is $\mathcal{T}_{\mathcal{S}} = \mathcal{T}_{\text{std}}$, and is $\mathcal{H}_{\mathcal{S}} = \mathcal{H}_{\text{std}}$?

We will answer this question for two different natural choices of $\mathcal{S}$. To understand these choices, we need several definitions.

Let $1 \leqslant k \leqslant n$ and let $r \geqslant 1$. Suppose V is an open subset of $\mathbb{R}^k$ and $f = (f_1, \ldots, f_n): V \to \mathbb{R}^n$ is a map. Recall that f is a C^r map if at each point of V all the partial derivatives of the f_i's of order $\leqslant r$ exist and are continuous. For each $x \in V$, let $f'(x)$ denote the $n \times k$ matrix whose (i, j)th entry is the first order partial derivative $(\partial f_i/\partial x_j)(x)$. f is *regular* if for every $x \in V$, the $n \times k$ matrix $f'(x)$ exists and is of rank k. f is a C^r *regular embedding* if f is a C^r regular topological embedding.

A subset M of $\mathbb{R}^n$ is a C^r *regular k-manifold* if for each $x \in M$, there is an open subset V of $\mathbb{R}^k$ and a C^r regular embedding $e: V \to \mathbb{R}^n$ such that $e(V)$ is a neighborhood of x in M.

A subset S of $\mathbb{R}^n$ is *smooth* if each point of S has a neighborhood U in $\mathbb{R}^n$ with the property that there is an $r \geqslant 1$ and a C^{r+1} map $f: U \to \mathbb{R}$ such that $S \cap U \subset f^{-1}(0)$ and f has a nonzero partial derivative of order $\leqslant r$ at each point of U. For example, the zero set of every nonconstant polynomial is a smooth set. Also, for $1 \leqslant k < n$, every C^2 regular k-manifold in $\mathbb{R}^n$ is a smooth set. A proof of this fact is sketched in the appendix.

We now formulate two theorems which answer our version of Laback's question for two choices of $\mathcal{S}$. In Theorem 1, $\mathcal{S}$ is required to include a class of subsets of $\mathbb{R}^n$ which forces $\mathcal{T}_{\mathcal{S}}$ to equal $\mathcal{T}_{\text{std}}$, and which is the smallest natural class with this property that the author could imagine. In Theorem 2, $\mathcal{S}$ is chosen to be a class of subsets of $\mathbb{R}^n$ for which $\mathcal{T}_{\mathcal{S}}$ fails to equal $\mathcal{T}_{\text{std}}$, and which is the largest natural class with this property that the author could imagine.

Theorem 1. *If $\mathcal{S}$ is any collection of subsets of $\mathbb{R}^n$ which contains all C^1 regular 1-manifolds, then $\mathcal{T}_{\mathcal{S}} = \mathcal{T}_{\text{std}}$ and, hence, $\mathcal{H}_{\mathcal{S}} = \mathcal{H}_{\text{std}}$.*

Theorem 2. *If $\mathcal{S}$ is the collection of all smooth subsets of $\mathbb{R}^n$, then $\mathcal{T}_{\mathcal{S}}$ is strictly larger than $\mathcal{T}_{\text{std}}$ and $\mathcal{H}_{\mathcal{S}}$ is strictly smaller than $\mathcal{H}_{\text{std}}$.*

Results similar to Theorem 1 have been obtained independently by C. Cooper in his 1990 University of Oklahoma Ph.D. thesis, and F. Gressl of Graz Technical University (unpublished).

The paper [2] investigates some related questions. The techniques developed here to prove Theorem 2 also allow us to generalize a construction in [2] and thereby to answer Question 3 of that paper. The result of our generalized construction is described in the next theorem.

Theorem 3. *There is a countable dense subset Z of $\mathbb{R}^n$ and an injective function $f:\mathbb{R}^n \to \mathbb{R}^n$ which is discontinuous at each point of Z, continuous at each point of $\mathbb{R}^n - Z$, and whose restriction to each smooth subset of $\mathbb{R}^n$ is continuous.*

Our proofs of Theorems 2 and 3 rely on the following technical theorem. Recall that an arc in $\mathbb{R}^n$ is *tame* if there is a homeomorphism of $\mathbb{R}^n$ which carries the arc to a straight line segment.

Theorem 4. *In $\mathbb{R}^n$, there is a tame arc A with endpoint 0 and an open set V containing $A-\{0\}$ with the property that if S is any smooth set containing 0, then $0 \notin \mathrm{cl}(S \cap V)$.*

The reader may wish to refer to [1] where related results for $\mathbb{R}^2$ are obtained. In [1], a different point of view is adopted, which leads to results that are not exactly parallel to those proved here. However, there is a strong similarity between the techniques used here and in [1].

The following notation is used at several points in this paper. For $x=(x_1,\ldots,x_n)$ and $y=(y_1,\ldots,y_n)\in\mathbb{R}^n$, let $x\cdot y$ denote the *dot product* of x and y, and let $|x|$ denote the *Euclidean norm* of x; thus, $x\cdot y=\sum x_iy_i$ and $|x|=(x\cdot x)^{1/2}$.

2. The proof of Theorem 1

It follows immediately from the definition of $\mathcal{T}_{\mathcal{S}}$ that $\mathcal{T}_{\mathrm{std}}\subset\mathcal{T}_{\mathcal{S}}$. Let $U\in\mathcal{T}_{\mathcal{S}}$. It suffices to prove that $U\in\mathcal{T}_{\mathrm{std}}$. Assume $U\notin\mathcal{T}_{\mathrm{std}}$. We will derive a contradiction.

Since $U\notin\mathcal{T}_{\mathrm{std}}$, then there is a sequence $\{x_n\}$ in $\mathbb{R}^n-U$ which converges (standardly) to a point $y\in U$. No element of $\mathcal{S}$ can contain both y and a subsequence of $\{x_n\}$. For if $y\in S\in\mathcal{S}$, then $U\cap S$ is a standard neighborhood of y in S. So any subsequence of $\{x_n\}$ that lies in S would eventually enter $U\cap S$, contradicting the fact that $\{x_n\}$ lies in $\mathbb{R}^n-U$. Now our strategy for reaching a contradiction is clear: we will construct an element of $\mathcal{S}$ which contains y and a subsequence of $\{x_n\}$.

For each $n\geq 1$, set $u_n=(x_n-y)/|x_n-y|$. By passing to a subsequence, we can assume that $\{u_n\}$ converges to a point $v\in\mathbb{R}^n$. Clearly $|v|=1$. For each $n\geq1$, set $t_n=(x_n-y)\cdot v$. By passing to a subsequence, we can assume that $\{t_n\}$ is a sequence of positive real numbers converging to 0 such that $t_{n+1}<\frac{1}{3}t_n$ for each $n\geq1$. For each $n\geq1$, set $J_n=[(\frac{1}{2})t_n,(\frac{3}{2})t_n]$. Then $J_m\cap J_n=\emptyset$ for $m\neq n$. For each $n\geq1$, set $w_n=x_n-y-t_nv$. Then $w_n\to0$ and $w_n/t_n\to0$.

We now define a C^1 regular embedding $\alpha:\mathbb{R}\to\mathbb{R}^n$ which passes through y and $\{x_n\}$. First let $\eta:\mathbb{R}\to[0,\infty)$ be a C^∞ map such that $\eta((-\infty,0]\cup[1,\infty))=\{0\}$ and

$\eta(\frac{1}{2})=1$. Define $\alpha:\mathbb{R}\to\mathbb{R}^n$ by $\alpha(t)=y+tv+\sum_{n\geqslant 1}\eta((t/t_n)-(\frac{1}{2}))w_n$. Thus, $\alpha(t)=y+tv+\eta((t/t_n)-(\frac{1}{2}))w_n$ if $t\in J_n$ for some $n\geqslant 1$, and $\alpha(t)=y+tv$ otherwise. So α is straight line perturbed by a sequence of bumps.

α passes through y and $\{x_n\}$, because $\alpha(0)=y$ and $\alpha(t_n)=x_n$ for $n\geqslant 1$. Clearly, α is C^∞ except possibly at $t=0$. Since η is bounded and $w_n\to 0$, then $\alpha(t)\to y=\alpha(0)$ as $t\to 0$. Hence, α is continuous at $t=0$ (as well as at all other values of t). Furthermore, α is a topological embedding because $(\alpha(t)-y)\cdot v=t$ for every $t\in\mathbb{R}$. Observe that $(\alpha(t)-y)/t=v+\eta((t/t_n)-(\frac{1}{2}))(w_n/t)$ if $t\in J_n$ for some $n\geqslant 1$, and $(\alpha(t)-y)/t=v$ otherwise. Since η is bounded, $w_n/t\leqslant 2w_n/t_n$ for $t\in J_n$, and $w_n/t_n\to 0$, it follows that $(\alpha(t)-y)/t\to v$ as $t\to 0$. So $\alpha'(0)$ exists and equals v. Observe that $\alpha'(t)=v+\eta'((t/t_n)-(\frac{1}{2}))(w_n/t_n)$ if $t\in J_n$ for some $n\geqslant 1$, and $\alpha'(t)=v$ otherwise. Since η' is bounded and $w_n/t_n\to 0$, it follows that $\alpha'(t)\to v=\alpha'(0)$ as $t\to 0$. Hence, α' is continuous at $t=0$ (as well as at all other values of t). This proves α is a C^1 map. Finally, $\alpha'(t)\cdot v=v\cdot v=1$ for each $t\in\mathbb{R}$, proving α is regular. We conclude that $\alpha:\mathbb{R}\to\mathbb{R}^n$ is a C^1 regular embedding which passes through y and $\{x_n\}$. So $\alpha(\mathbb{R})$ is a C^1 regular 1-manifold in $\mathbb{R}^n$.

Since $\alpha(\mathbb{R})$ is an element of $\mathscr{S}$ which contains y and a subsequence of the original $\{x_n\}$, we have reached the desired contradiction. □

3. The proof of Theorem 4

Our proof of Theorem 4 uses the following notation. Set $\omega=\{0,1,2,\dots\}$. For $a=(a_1,\dots,a_n)\in\omega^n$, set $\|a\|=\sum_{1\leqslant i\leqslant n}a_i$ and set $a!=\prod_{1\leqslant i\leqslant n}(a_i!)$. For $x=(x_1,\dots,x_n)\in\mathbb{R}^n$ and $a=(a_1,\dots,a_n)\in\omega^n$, set $x^a=\prod_{1\leqslant i\leqslant n}x_i^{a_i}$. If X is a set, $\varphi=(\varphi_1,\dots,\varphi_n):X\to\mathbb{R}^n$ is a function, and $a=(a_1,\dots,a_n)\in\omega^n$, then define the function $\varphi^a:X\to\mathbb{R}$ by $\varphi^a(x)=(\varphi(x))^a=\prod_{1\leqslant i\leqslant n}(\varphi_i(x))^{a_i}$ for $x\in X$. For $a=(a_1,\dots,a_n)\in\omega^n$, if U is an open subset of $\mathbb{R}^n$ and $f:U\to\mathbb{R}$ is a sufficiently differentiable map, then for every $p\in U$ set

$$f^{(a)}(p)=\frac{\partial^{\|a\|}f}{\partial x_1^{a_1}\dots\partial x_n^{a_n}}(p).$$

Let $r\geqslant 1$, and let U be an open subset of $\mathbb{R}^n$. A function $f:U\to\mathbb{R}$ is a C^r map if for each $a\in\omega^n$ with $\|a\|\leqslant r, f^{(a)}(p)$ exists for every $p\in U$ and $f^{(a)}:U\to\mathbb{R}$ is continuous. Let $C^r(U)$ denote the set of all C^r maps from U to $\mathbb{R}$.

Let $r\geqslant 1$, let U be an open subset of $\mathbb{R}^n$, let $f\in C^r(U)$, and let $p\in U$. The *degree r Taylor polynomial of f at p* is

$$T_p^rf(x)=\sum_{\substack{a\in\omega^n\\ \|a\|\leqslant r}}\frac{1}{a!}f^{(a)}(p)x^a$$

for $x\in\mathbb{R}^n$.

Our notation allows us to state:

A version of Taylor's formula. *Let $r \geq 1$, let U be an open subset of $\mathbb{R}^n$, let $f \in C^{r+1}(U)$, and let $p \in U$. If $x \in \mathbb{R}^n$ such that U contains the straight line segment from p to $p+x$, then there is a $\theta \in (0,1)$ such that*

$$f(p+x) = T_p^r f(x) + \sum_{\substack{a \in \omega^n \\ \|a\| = r+1}} \frac{1}{a!} f^{(a)}(p+\theta x) x^a.$$

In the appendix, we indicate how this formula is derived from a version of Taylor's formula commonly found in advanced calculus texts.

Observe that if $r \geq 1$, U is an open subset of $\mathbb{R}^n$, and $f \in C^r(U)$, then f has a nonzero partial derivative of order $\leq r$ at p if and only if $T_p^r f(x) \neq 0$. Using this observation, we restate the definition of *smooth*. A subset S of $\mathbb{R}^n$ is *smooth* if each point of S has a neighborhood U in $\mathbb{R}^n$ with the property that there is an $r \geq 1$ and an $f \in C^{r+1}(U)$ such that $S \cap U \subset f^{-1}(0)$ and $T_p^r f \neq 0$ for every $p \in U$.

Next, we define a linear order $<$ on ω^n. For $a, b \in \omega^n$, we declare $a < b$ if either (1) $\|a\| < \|b\|$ or (2) $\|a\| = \|b\|$ and there is a k such that $1 \leq k \leq n$, $a_i = b_i$ for $1 \leq i < k$, and $a_k < b_k$. We observe that $<$ is a well ordering of ω^n, because for each $a \in \omega^n$, $\{b \in \omega^n : b < a\}$ is a finite set.

Our proof of Theorem 4 depends on the following lemma.

Lemma. *There are order preserving homeomorphisms $\varphi_1, \ldots, \varphi_n : [0,1] \to [0,1]$ with the following property. Define the embedding $\varphi : [0,1] \to \mathbb{R}^n$ by $\varphi = (\varphi_1, \ldots, \varphi_n)$. If $a, b \in \omega^n$ and $a < b$, then*

$$\lim_{t \to 0} \frac{\varphi^b(t)}{\varphi^a(t)} = 0.$$

Proof of Lemma. We begin by defining the homeomorphism $\psi : [0,1] \to [0,1]$ by $\psi(0) = 0$ and $\psi(t) = \ln 2/(\ln 2 - \ln t)$ for $0 < t \leq 1$. By applying l'Hospital's rule to $\ln t / t^{-r}$, we find that $t^r \ln t \to 0$ as $t \to 0$ for any $r > 0$. It follows that for any $s > 0$, $t^{1/s}/\psi(t) \to 0$ as $t \to 0$. Thus, for any $s \geq 0$, $t/(\psi(t))^s \to 0$ as $t \to 0$.

Next define the homeomorphism $\psi_i : [0,1] \to [0,1]$ for each $i \geq 1$ by $\psi_1 = \psi$ and $\psi_i = \psi \circ \psi_{i-1}$ for $i > 1$. Then for $s \geq 0$ and $i \geq 1$, since $\psi_i(t) \to 0$ as $t \to 0$, and since $\psi_i(t)/(\psi_{i+1}(t))^s = \psi_i(t)/(\psi(\psi_i(t)))^s$, then the last line of the preceding paragraph implies that $\psi_i(t)/(\psi_{i+1}(t))^s \to 0$ as $t \to 0$.

Finally for each $i \geq 1$, define the homeomorphism $\varphi_i : [0,1] \to [0,1]$ by $\varphi_i(t) = t\psi_i(t)$. The embedding $\varphi : [0,1] \to \mathbb{R}^n$ is defined by $\varphi = (\varphi_1, \ldots, \varphi_n)$. Recall that if $a = (a_1, \ldots, a_n) \in \omega^n$, then $\varphi^a : [0,1] \to [0,1]$ is given by

$$\varphi^a(t) = (\varphi_1(t))^{a_1} \cdots (\varphi_n(t))^{a_n}.$$

Let $a, b \in \omega^n$ such that $a < b$. Then there is a finite sequence $a = c_0 < c_1 < \cdots < c_k = b$ in ω^n such that c_i is the immediate successor of c_{i-1} for $1 \leq i \leq k$. Since

$$\frac{\varphi^b(t)}{\varphi^a(t)} = \prod_{1 \leq i \leq k} \frac{\varphi^{c_i}(t)}{\varphi^{c_{i-1}}(t)},$$

then it clearly suffices to consider the situation in which b is the immediate successor of a.

There are two cases.

Case 1: $\|a\|<\|b\|$. In this case $a=(r,0,\ldots,0,0)$ and $b=(0,0,\ldots,0,r+1)$. So

$$\frac{\varphi^b(t)}{\varphi^a(t)}=\frac{(\varphi_n(t))^{r+1}}{(\varphi_1(t))^r}=(\psi_n(t))^{r+1}\left[\frac{t}{(\psi(t))^r}\right].$$

It now follows from our earlier observations that $\varphi^b(t)/\varphi^a(t)\to 0$ as $t\to 0$.

Case 2: $\|a\|=\|b\|$. In this case there is a k, $1\leq k<n$, such that $a_i=b_i$ for $1\leq i<k$, $a=(a_1,\ldots,a_{k-1},r,s,0,\ldots,0,0)$ and $b=(a_1,\ldots,a_{k-1},r+1,0,0,\ldots,0,s-1)$. So

$$\frac{\varphi^b(t)}{\varphi^a(t)}=\frac{(\varphi_k(t))^{r+1}(\varphi_n(t))^{s-1}}{(\varphi_k(t))^r(\varphi_{k+1}(t))^s}=\left[\frac{\psi_k(t)}{(\psi_{k+1}(t))^s}\right](\psi_n(t))^{s-1}.$$

Again our earlier observations imply that $\varphi^b(t)/\varphi^a(t)\to 0$ as $t\to 0$. □

We now prove Theorem 4. We define the arc A in $\mathbb{R}^n$ by $A=\varphi([0,1])$, where $\varphi:[0,1]\to\mathbb{R}^n$ is the embedding of the preceding lemma. Then $0=\varphi(0)\in\partial A$. To see that A is tame, observe that for $1\leq i\leq n$, the homeomorphism $\varphi_i:[0,1]\to[0,1]$ extends to a homeomorphism $\Phi_i:\mathbb{R}\to\mathbb{R}$ such that $\Phi_i=\text{id}$ on $(-\infty,0]\cup[1,\infty)$. Define the homeomorphism $h:\mathbb{R}^n\to\mathbb{R}^n$ by $h(x)=(\Phi_1(x_1),\ldots,\Phi_n(x_n))$ for $x=(x_1,\ldots,x_n)\in\mathbb{R}^n$, and note that h carries the straight line segment $\{(t,\ldots,t): 0\leq t\leq 1\}$ onto A.

For each $t\in(0,1]$, define the neighborhood $V(t)$ of $\varphi(t)$ in $\mathbb{R}^n$ by

$$V(t)=\{(x_1,\ldots,x_n)\in\mathbb{R}^n: 2^{-1}\varphi_i(t)<x_i<2\varphi_i(t)\text{ for }1\leq i\leq n\}.$$

Observe that if $t\in(0,1]$, $x=(x_1,\ldots,x_n)\in V(t)$, and $a\in\omega^n$, then $2^{-\|a\|}\varphi^a(t)<x^a<2^{\|a\|}\varphi^a(t)$. Next define $V=\bigcup_{0<t\leq 1}V(t)$. Then V is an open subset of $\mathbb{R}^n$ containing $A-\{0\}$.

Let S be a smooth set containing 0. We shall prove that $0\notin\text{cl}(V\cap S)$. For assume otherwise. Then there is a sequence $\{x_k\}$ in $V\cap S$ that converges to 0. For each $k\geq 1$, there is a $t_k\in(0,1]$ such that $x_k\in V(t_k)$. It follows from the way in which the $V(t)$ are defined that $\{t_k\}$ must converge to 0.

Since S is a smooth set and $0\in S$, there is a neighborhood U of 0 in $\mathbb{R}^n$, an $r\geq 1$, and an $f\in C^{r+1}(U)$ such that $S\cap U\subset f^{-1}(0)$ and $T_p^rf\neq 0$ for every $p\in U$. We will argue that $T_0^rf=0$, and thereby reach a contradiction.

We can assume that $\{x_k\}$ lies in U, and that for each $k\geq 1$, U contains the straight line segment from 0 to x_k. For each $k\geq 1, f(x_k)=0$ because $x_k\in S\cap U$. So, for each $k\geq 1$, the Taylor formula for $f(x_k)$ takes the form

$$0=T_0^rf(x_k)+\sum_{\substack{a\in\omega^n\\ \|a\|=r+1}}\frac{1}{a!}f^{(a)}(\theta_kx_k)(x_k)^a$$

for some $\theta_k\in(0,1)$.

To make the right side of this formula more uniform, we define $z_{s,k} \in \mathbb{R}^n$ for $0 \leq s \leq r+1$ and $k \geq 1$ as follows. For $k \geq 1$, set $z_{s,k} = 0$ if $0 \leq s \leq r$, and set $z_{r+1,k} = \theta_k x_k$. Observe that for any fixed s between 0 and $r+1$, $\lim_{k\to\infty} z_{s,k} = 0$. Now the two terms on the right side of this formula can be absorbed into a single summation in which $\|a\|$ runs from 0 to $r+1$. For each $k \geq 1$, we rewrite the Taylor formula for $f(x_k)$ as

$$0 = \sum_{\substack{a \in \omega^n \\ \|a\| \leq r+1}} \frac{1}{a!} f^{(a)}(z_{\|a\|,k})(x_k)^a.$$

We now begin the inductive proof that $T_0^r f = 0$. The first term of $T_0^r f$ is $f^{(0)}(0) = f(0)$. $f(0) = 0$ because $0 \in S \cap U$. So $f^{(0)}(0) = 0$.

Next let $a \in \omega^n$ such that $0 < \|a\| \leq r$, and inductively assume that if $b \in \omega^n$ and $b < a$, then $f^{(b)}(0) = 0$. Then for each $k \geq 1$, the Taylor formula for $f(x_k)$ takes the form

$$0 = \sum_{\substack{b \in \omega^n \\ \|b\| \leq r+1 \\ a \leq b}} \frac{1}{b!} f^{(b)}(z_{\|b\|,k})(x_k)^b.$$

By passing to a subsequence of $\{x_k\}$, we can assume that for each $b \in \omega^n$ with $\|b\| \leq r+1$, the sequence $\{f^{(b)}(z_{\|b\|,k})\}$ does not take on both positive and negative values. Then for each $b \in \omega^n$ with $\|b\| \leq r+1$, set $\varepsilon(b) = +1$ or -1 depending on whether $\{f^{(b)}(z_{\|b\|,k})\}$ is nonnegative or nonpositive.

For each $k \geq 1$, since $x_k \in V(t_k)$, then for each $b \in \omega^n$ with $\|b\| \leq r+1$, we have the inequality

$$2^{-\|b\|} \varphi^b(t_k) < (x_k)^b < 2^{\|b\|} \varphi^b(t_k).$$

Multiplying this inequality by $f^{(b)}(z_{\|b\|,k})$ yields the inequality

$$2^{-\varepsilon(b)\|b\|} f^{(b)}(z_{\|b\|,k}) \varphi^b(t_k) \leq f^{(b)}(z_{\|b\|,k})(x_k)^b$$
$$\leq 2^{\varepsilon(b)\|b\|} f^{(b)}(z_{\|b\|,k}) \varphi^b(t_k).$$

Dividing the preceding inequality by $b!$, summing over all $b \in \omega^n$ with $\|b\| \leq r+1$ and $a \leq b$, and recognizing the middle summation as a version of the Taylor formula for $f(x_k)$, yields the inequality

$$\sum_{\substack{b \in \omega^n \\ \|b\| \leq r+1 \\ a \leq b}} 2^{-\varepsilon(b)\|b\|} \frac{1}{b!} f^{(b)}(z_{\|b\|,k}) \varphi^b(t_k) \leq 0 \leq \sum_{\substack{b \in \omega^n \\ \|b\| \leq r+1 \\ a \leq b}} 2^{\varepsilon(b)\|b\|} \frac{1}{b!} f^{(b)}(z_{\|b\|,k}) \varphi^b(t_k).$$

We divide the preceding inequality by $\varphi^a(t_k)$, to obtain

$$\sum_{\substack{b \in \omega^n \\ \|b\| \leq r+1 \\ a \leq b}} 2^{-\varepsilon(b)\|b\|} \frac{1}{b!} f^{(b)}(z_{\|b\|,k}) \frac{\varphi^b(t_k)}{\varphi^a(t_k)} \leq 0 \leq \sum_{\substack{b \in \omega^n \\ \|b\| \leq r+1 \\ a \leq b}} 2^{\varepsilon(b)\|b\|} \frac{1}{b!} f^{(b)}(z_{\|b\|,k}) \frac{\varphi^b(t_k)}{\varphi^a(t_k)}.$$

Now we let $k\to\infty$ in this inequality. Then $t_k\to 0$. So the above lemma implies that if $b\in\omega^n$, $\|b\|\leq r+1$, and $a<b$, then $\varphi^b(t_k)/\varphi^a(t_k)\to 0$. Also if $b\in\omega^n$ and $\|b\|\leq r+1$, then $f^{(b)}(z_{\|b\|,k})\to f^{(b)}(0)$ because $z_{\|b\|,k}\to 0$. Thus, all the terms of the summations vanish except the $b=a$ terms. $z_{\|a\|,k}=0$ because $\|a\|\leq r$. So we are left with the inequality

$$2^{-\varepsilon(a)\|a\|}\frac{1}{a!}f^{(a)}(0)\leq 0\leq 2^{\varepsilon(a)\|a\|}\frac{1}{a!}f^{(a)}(0).$$

Since $2^{\pm\varepsilon(a)\|a\|}/a!>0$, we conclude that $f^{(a)}(0)=0$.

It now follows inductively that $f^{(a)}(0)=0$ for each $a\in\omega^n$ such that $\|a\|\leq r$. Therefore, $T_0^r f=0$. We have now reached the contradiction we sought. We conclude that $0\notin \mathrm{cl}\,(V\cap S)$. □

4. The proof of Theorem 2

Let $\mathcal{S}$ denote the collection of all smooth subsets of $\mathbb{R}^n$. It follows immediately from the definition of $\mathcal{T}_{\mathcal{S}}$ that $\mathcal{T}_{\mathrm{std}}\subset\mathcal{T}_{\mathcal{S}}$. Let A be the arc constructed in Theorem 4. $A-\{0\}$ is not a standard closed subset of $\mathbb{R}^n$. However, Theorem 4 implies that $A-\{0\}$ is a closed subset of $\mathbb{R}^n$ with respect to the topology $\mathcal{T}_{\mathcal{S}}$. So $(\mathbb{R}^n-A)\cup\{0\}\notin\mathcal{T}_{\mathrm{std}}$, but $(\mathbb{R}^n-A)\cup\{0\}\in\mathcal{T}_{\mathcal{S}}$. Therefore, $\mathcal{T}_{\mathcal{S}}$ is strictly larger than $\mathcal{T}_{\mathrm{std}}$.

Since A is a tame arc, there is a (standard) homeomorphism $h:\mathbb{R}^n\to\mathbb{R}^n$ such that $h(A)$ is a straight line segment. Thus, $h\in\mathcal{H}_{\mathrm{std}}$. Since every straight line in $\mathbb{R}^n$ is a smooth set, and every subset of a smooth set is smooth, then $h(A)\in\mathcal{S}$. Hence, $\mathcal{T}_{\mathcal{S}}$ restricts to the standard topology on $h(A)$. With respect to the standard topology on $h(A)$, $h(0)$ is a limit point of $h(A-\{0\})$. So $h(A-\{0\})$ is not a closed subset of $h(A)$ with respect to either $\mathcal{T}_{\mathrm{std}}$ or $\mathcal{T}_{\mathcal{S}}$. Consequently, with respect to $\mathcal{T}_{\mathcal{S}}$, $A-\{0\}$ is a closed subset of $\mathbb{R}^n$ but $h(A-\{0\})$ is not. We conclude that $h\notin\mathcal{H}_{\mathcal{S}}$. This proves $\mathcal{H}_{\mathcal{S}}\neq\mathcal{H}_{\mathrm{std}}$.

Before proving $\mathcal{H}_{\mathcal{S}}\subset\mathcal{H}_{\mathrm{std}}$, we make two observations. Let $h\in\mathcal{H}_{\mathcal{S}}$.

(1) If $S\in\mathcal{S}$, then $h|S:S\to\mathbb{R}^n$ is continuous (in the standard sense).

(2) If U is a connected open subset of $\mathbb{R}^n$, then $h(U)$ is connected.

(1) follows because $\mathcal{T}_{\mathcal{S}}$ restricts to the standard topology on S and $\mathcal{T}_{\mathrm{std}}\subset\mathcal{T}_{\mathcal{S}}$. (1) implies that if J is a straight line segment in $\mathbb{R}^n$, then $h(J)$ is connected. To prove (2), let x and $y\in U$. Then x and y are joined by a piecewise linear path J in U. Since h maps each straight piece of J to a connected set, then $h(J)$ is a connected subset of $h(U)$ joining $h(x)$ to $h(y)$. This proves $h(U)$ is connected.

We now prove $\mathcal{H}_{\mathcal{S}}\subset\mathcal{H}_{\mathrm{std}}$. Let $h\in\mathcal{H}_{\mathcal{S}}$. Suppose $U\in\mathcal{T}_{\mathrm{std}}$. We will prove $h(U)\in\mathcal{T}_{\mathrm{std}}$. Let $x\in U$. Choose $r>0$ so that $\{y\in\mathbb{R}^n: |x-y|\leq r\}\subset U$. Let $S=\{y\in\mathbb{R}^n: |x-y|=r\}$, let $V=\{y\in\mathbb{R}^n: |x-y|<r\}$ and let $W=\{y\in\mathbb{R}^n: |x-y|>r\}$. Since S is a compact smooth set, then observation (1) implies $h|S$ is an embedding. So $h(S)$ is an $(n-1)$-sphere in $\mathbb{R}^n$. The Jordan Separation Theorem now implies that

$\mathbb{R}^n - h(S)$ has precisely two components. Moreover, these components are open subsets of $\mathbb{R}^n$. Observation (2) implies that $h(V)$ and $h(W)$ are connected. Also, since $h:\mathbb{R}^n \to \mathbb{R}^n$ is a bijection, then $\mathbb{R}^n - h(S) = h(V) \cup h(W)$ and $h(V) \cap h(W) = \emptyset$. It follows that $h(V)$ and $h(W)$ are the two components of $\mathbb{R}^n - h(S)$. Hence, $h(V)$ is an open subset of $\mathbb{R}^n$ such that $h(x) \in h(V) \subset h(U)$. This proves $h(U) \in \mathcal{T}_{\text{std}}$. We can similarly prove that if $U \in \mathcal{T}_{\text{std}}$, then $h^{-1}(U) \in \mathcal{T}_{\text{std}}$. It follows that $h \in \mathcal{H}_{\text{std}}$. We have now proved that $\mathcal{H}_{\mathcal{S}} \subset \mathcal{H}_{\text{std}}$.

We conclude that $\mathcal{H}_{\mathcal{S}}$ is strictly smaller than $\mathcal{H}_{\text{std}}$. □

5. The proof of Theorem 3

We refer the reader to [2, Section 4] for a 2-dimensional version of this construction.

Let A be the tame arc and V the open set in $\mathbb{R}^n$ constructed in Theorem 4. For each $p \in \mathbb{R}^n$, let $A(p) = A + p = \{x + p\colon x \in A\}$ and let $V(p) = V + p = \{x + p\colon x \in V\}$. Since translation takes smooth sets to smooth sets, it follows that if S is a smooth set and $p \in S$, then $p \notin \text{cl}(S \cap V(p))$.

The desired function $f:\mathbb{R}^n \to \mathbb{R}^n$ arises as a composition $f = g \circ h$ where $g, h:\mathbb{R}^n \to \mathbb{R}^n$ are functions satisfying the following four conditions.

(1) Y is a countable dense subset of $\mathbb{R}^n$.

(2) $g:\mathbb{R}^n \to \mathbb{R}^n$ is an injective function which is discontinuous at each point of Y and continuous at each point of $\mathbb{R}^n - Y$.

(3) For each $y \in Y$, there is an open subset $W(y)$ of $\mathbb{R}^n$ such that if $\{w_k\}$ is a sequence in $\mathbb{R}^n - W(y)$ that converges to y, then $\{g(w_k)\}$ converges to $g(y)$.

(4) $h:\mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism such that for each $y \in Y$, $h(V(h^{-1}(y)))$ contains $W(y)$.

Assume for the moment that we have functions $g, h:\mathbb{R}^n \to \mathbb{R}^n$ satisfying conditions (1)-(4). Set $Z = h^{-1}(Y)$ and $f = g \circ h$. Then clearly Z is a countable dense subset of $\mathbb{R}^n$, and f is an injective function which is discontinuous at each point of Z and continuous at each point of $\mathbb{R}^n - Z$. Let S be a smooth subset of $\mathbb{R}^n$. Clearly, $f|S$ is continuous at each point of $S - Z$. Suppose $z \in S \cap Z$. Let $\{w_k\}$ be a sequence in S that converges to z. Since $z \notin \text{cl}(S \cap V(z))$, then we can assume that $\{w_k\}$ avoids $V(z)$. Now, $h(z) \in Y$, and $\{h(w_k)\}$ converges to $h(z)$ and avoids $W(h(z))$. So, by condition (3), $\{g(h(w_k))\}$ converges to $g(h(z))$. Hence, $\{f(w_k)\}$ converges to $f(z)$. This proves $f|S$ is continuous at z. We conclude that $f|S$ is continuous. It remains to construct the functions g and h.

We now construct g. Let $\{x_i : i \geq 1\}$ be a countable dense subset of $\mathbb{R}^n$. Let $v \in \mathbb{R}^n$ such that $|v| = 1$ and $v \neq (x_i - x_j)/|x_i - x_j|$ for all $i \neq j$. For each $i \geq 1$, define $C_i = \{x_i + tv\colon -1/i \leq t \leq 0\}$, $D_i = \{x_i + tv\colon 0 \leq t \leq 1/i\}$, $E_i = \{x_i + tv\colon 1/i \leq t \leq 2/i\}$, and $F_i = C_i \cup D_i \cup E_i$. Then the collection $\mathcal{D} = \{D_i : i \geq 1\} \cup \{\{x\}\colon x \in \mathbb{R}^n - \bigcup_{i \geq 1} D_i\}$ is an upper semicontinuous decomposition of $\mathbb{R}^n$ into a null sequence of tame arcs and points. Hence, $\mathcal{D}$ is shrinkable [3, p. 56]. So there is a closed onto map $\pi:\mathbb{R}^n \to \mathbb{R}^n$ such that $\{\pi^{-1}(x)\colon x \in \mathbb{R}^n\} = \mathcal{D}$.

Let $i \geqslant 1$.Note that $\pi(F_i)$ is an arc. We will prove that $\pi(F_i)$ is tame. First, observe that the decomposition $\mathscr{E} = \{D_j : i \neq j \geqslant 1\} \cup \{\{x\}: x \in \mathbb{R}^n - \bigcup_{i \neq j \geqslant 1} D_j\}$ is strongly shrinkable [3, p. 56]. Hence, there is a closed onto map $\rho : \mathbb{R}^n \to \mathbb{R}^n$ such that $\{\rho^{-1}(x): x \in \mathbb{R}^n\} = \mathscr{E}$ and $\rho | F_i = \mathrm{id}$. Also, clearly, there is a closed onto map $\sigma : \mathbb{R}^n \to \mathbb{R}^n$ such that $\sigma(D_i) = \{x_i\}$, $\sigma(F_i) = F_i$, and $\{\sigma^{-1}(x): x \in \mathbb{R}^n\} = \{D_i\} \cup \{\{x\}: x \in \mathbb{R}^n - D_i\}$. Hence, $\sigma \circ \rho : \mathbb{R}^n \to \mathbb{R}^n$ is a closed onto map such that $\{(\sigma \circ \rho)^{-1}(x): x \in \mathbb{R}^n\} = \mathscr{D}$ and $\sigma \circ \rho(F_i) = F_i$. Since, $\{\pi^{-1}(x): x \in \mathbb{R}^n\} = \{(\sigma \circ \rho)^{-1}(x): x \in \mathbb{R}^n\}$, then there is a homeomorphism $\tau : \mathbb{R}^n \to \mathbb{R}^n$ such that $\tau \circ \pi = \sigma \circ \rho$. So $\tau(\pi(F_i)) = \sigma \circ \rho(F_i) = F_i$. This proves $\pi(F_i)$ is tame.

For each $i \geqslant 1$, set $y_i = \pi(D_i) = \pi(x_i)$, and set $Y = \{y_i : i \geqslant 1\}$. Since $\{x_i : i \geqslant 1\}$ is dense in $\mathbb{R}^n$, then so is Y. Define $g : \mathbb{R}^n \to \mathbb{R}^n$ by $g(x) = \pi^{-1}(x)$ if $x \in \mathbb{R}^n - Y$ and $g(y_i) = x_i$ for $i \geqslant 1$. Then $\pi \circ g = \mathrm{id}$. So g is injective.

To prove that g is continuous at each $x \in \mathbb{R}^n - Y$, let N be a neighborhood of $g(x) = \pi^{-1}(x)$ in $\mathbb{R}^n$. Since π is a closed map, there is a neighborhood M of x in $\mathbb{R}^n$ such that $\pi^{-1}(M) \subset N$. Since $g(M) \subset \pi^{-1}(M)$, then $g(M) \subset N$.

To prove that for each $i \geqslant 1$, g is discontinuous at y_i, fix $i \geqslant 1$, and set $w_k = \pi(x_i + (1/i + 1/k)v)$ for each $k \geqslant 1$. Then $\{w_k\}$ is a sequence in $\mathbb{R}^n - Y$ that converges to $\pi(x_i + (1/i)v) = y_i$. However, $\{g(w_k)\}$ does not converge to $g(y_i)$, because $\{g(w_k)\}$ converges to $x_i + (1/i)v$ and $g(y_i) = x_i$.

We remark that at this point we have verified the first two of conditions (1)–(4) stated at the beginning of this proof.

Next, for each $i \geqslant 1$, we find an open subset P_i of $\mathbb{R}^n$ such that

(5) $\pi(E_i) - \{y_i\} \subset P_i$,

(6) $\mathrm{cl}(P_i) \cap \pi(C_i) = \{y_i\}$, and

(7) if $\{w_k\}$ is a sequence in $\mathbb{R}^n - P_i$ that converges to y_i, then $\{g(w_k)\}$ converges to $g(y_i)$.

Fix $i \geqslant 1$. If U is an open subset of $\mathbb{R}^n$, let $U^* = \bigcup\{D \in \mathscr{D}: D \subset U\}$; then U^* and $\pi(U^*)$ are open subsets of $\mathbb{R}^n$, because π is a closed map. Let Q and R be open subsets of $\mathbb{R}^n$ such that $C_i - \{x_i\} \subset Q$, $(D_i \cup E_i) - \{x_i\} \subset R$, and $Q \cap R = \emptyset$. Then Q^* and R^* are open subsets of $\mathbb{R}^n$ such that $C_i - \{x_i\} \subset Q^*$ and $E_i - \{x_i + (1/i)v\} \subset R^*$. It follows that $\pi(Q^*)$ and $\pi(R^*)$ are disjoint open subsets of $\mathbb{R}^n$ such that $\pi(C_i) - \{y_i\} \subset \pi(Q^*)$ and $\pi(E_i) - \{y_i\} \subset \pi(R^*)$. Consequently, $\pi(C_i) \cap \mathrm{cl}(\pi(R^*)) = \{y_i\}$. Set $P_i = \pi(R^*)$. Then P_i is an open subset of $\mathbb{R}^n$ such that $\pi(E_i) - \{y_i\} \subset P_i$ and $\mathrm{cl}(P_i) \cap \pi(C_i) = \{y_i\}$.

Suppose $\{w_k\}$ is a sequence in $\mathbb{R}^n - P_i$ that converges to y_i. We must prove that $\{g(w_k)\}$ converges to $g(y_i) = x_i$. Let N be a neighborhood of x_i in $\mathbb{R}^n$. The collection $\mathscr{E} = \{D_j : i \neq j \geqslant 1\} \cup \{\{x\}: x \in \mathbb{R}^n - \bigcup_{i \neq j \geqslant 1} D_j\}$ is an upper semicontinuous decomposition of $\mathbb{R}^n$. If U is an open subset of $\mathbb{R}^n$, set $U^{\#} = \bigcup\{E \in \mathscr{E}: E \subset U\}$; then $U^{\#}$ is an open subset of $\mathbb{R}^n$. $D_i - \{x_i\} \subset R^*$, because $D_i - \{x_i\} \subset R$; and $x_i \in N^{\#}$. So $N^{\#} \cup R^{\#}$ is a neighborhood of $D_i = \pi^{-1}(y_i)$ in $\mathbb{R}^n$. Since π is a closed map, there is a neighborhood M of y_i in $\mathbb{R}^n$ such that $\pi^{-1}(M) \subset N^{\#} \cup Q^{\#}$. Since $\{w_k\}$ converges to y_i, then there is a $K \geqslant 1$ such that $w_k \in M$ for $k \geqslant K$. Let $k \geqslant K$. If $w_k = y_i$, then $g(w_k) = x_i \in N$. So suppose $w_k \neq y_i$. Then $\pi^{-1}(w_k) \in \mathscr{E}$. Since $w_k \notin P_i = \pi(R^{\#})$, then

$\pi^{-1}(w_k)\not\subset R$. So $\pi^{-1}(w_k)\cap R^{\#}=\emptyset$. Since $\pi^{-1}(w_k)\subset\pi^{-1}(M)\subset N^{\#}\cup R^{\#}$, it follows that $\pi^{-1}(w_k)\subset N^{\#}$. Since $g(w_k)\in\pi^{-1}(w_k)$ and $N^{\#}\subset N$, then $g(w_k)\in N$. This proves $\{g(w_k)\}$ converges to $x_i=g(y_i)$.

Observe that property (7) implies that if $i\geq 1$, M is a neighborhood of y_i in $\mathbb{R}^n$, and $\{w_k\}$ is a sequence in $\mathbb{R}^n-(P_i\cap M)$ that converges to y_i, then $\{g(w_k)\}$ converges to $g(y_i)$.

Recall that A is the tame arc and V is the open set in $\mathbb{R}^n$ provided by Theorem 4, 0 is an endpoint of A, and $A-\{0\}\subset V$. Let V' be an open subset of $\mathbb{R}^n$ such that $A-\{0\}\subset V'$ and cl $V'\subset\{0\}\cup V$. For each $p\in\mathbb{R}^n$, let $V'(p)=V'+p=\{x+p: x\in V'\}$.

We obtain the homeomorphism h of $\mathbb{R}^n$ as the limit of a sequence $\{h_i\}$ of homeomorphisms of $\mathbb{R}^n$. The sequence $\{h_i\}$ together with a sequence $\{M_i\}$ of open subsets of $\mathbb{R}^n$ are constructed inductively to satisfy the following four conditions.

(8) For each $x\in\mathbb{R}^n$, $|h_i(x)-h_{i+1}(x)|<2^{-i}$ and $|h_i^{-1}(x)-h_{i+1}^{-1}(x)|<2^{-i}$.

(9) For each $j\geq i$, $h_j^{-1}(y_i)=h_i^{-1}(y_i)$.

(10) $y_i\in M_i$.

(11) For each $j\geq i$, $h_j(V'(h_i^{-1}(y_i)))$ contains $\mathrm{cl}(P_i\cap M_i)-\{y_i\}$.

Assume for the moment that we have $\{h_i\}$ and $\{M_i\}$ satisfying conditions (8)-(11). Then condition (8) guarantees that $\{h_i\}$ converges uniformly to a map $h:\mathbb{R}^n\to\mathbb{R}^n$. Moreover, h is a homeomorphism, because the second inequality in (8) implies that $\{h_i^{-1}\}$ converges uniformly to h^{-1}. Next we verify conditions (3) and (4) (stated at the beginning of this proof). Condition (9) implies that $h^{-1}(y_i)=h_i^{-1}(y_i)$ for $i\geq 1$. For each $i\geq 1$, let $W(y_i)=P_i\cap M_i$. Now for each $i\geq 1$, property (7) implies that if $\{w_k\}$ is a sequence in $\mathbb{R}^n-W(y_i)$ that converges to y_i, then $\{g(w_k)\}$ converges to $g(y_i)$. Condition (11) and the observation that $h^{-1}(y_i)=h_i^{-1}(y_i)$ imply that $h_j^{-1}(P_i\cap M_i)\subset V'(h^{-1}(y_i))$ for $j\geq i$. Since $\{h_j^{-1}\}$ converges to h^{-1}, we deduce that $h^{-1}(P_i\cap M_i)\subset \mathrm{cl}(V'(h^{-1}(y_i)))$. Since $\mathrm{cl}(V'(h^{-1}(y_i)))\subset\{h^{-1}(y_i)\}\cup V(h^{-1}(y_i))$ and $y_i\notin P_i$, then $h^{-1}(P_i\cap M_i)\subset V(h^{-1}(y_i))$. We conclude that $h(V(h^{-1}(y_i)))$ contains $W(y_i)$.

It remains to construct $\{h_i\}$ and $\{M_i\}$. We begin this construction by setting $h_0=\mathrm{id}_{\mathbb{R}^n}$ and $M_0=\emptyset$. Let $i\geq 1$, and inductively assume we have h_j and M_j for $0\leq j<i$. Set $z=h_{i-1}^{-1}(y_i)$. Choose a neighborhood N of z in $\mathbb{R}^n$ such that $h_{i-1}^{-1}(y_j)\notin N$ for $1\leq j<i$, diam $N<2^{-i}$ and diam $h_{i-1}(N)<2^{-i}$, and such that for $1\leq j<i$, if N intersects $h_{i-1}^{-1}(\mathrm{cl}(P_j\cap M_j))$, then $N\subset V'(h_{i-1}^{-1}(y_j))$. Now recall that $A(z)$ and $h_{i-1}^{-1}(\pi(F_i))=h_{i-1}^{-1}(\pi(C_i))\cup h_{i-1}^{-1}(\pi(E_i))$ are tame arcs and z is an endpoint of $A(z)$, $h_{i-1}^{-1}(\pi(C_i))$, and $h_{i-1}^{-1}(\pi(E_i))$. Hence, there is a homeomorphism τ_1 of $\mathbb{R}^n$ which fixes z and takes $A(z)$ onto $h_{i-1}^{-1}(\pi(E_i))$. Using the Annulus Theorem ([8, 5] for dimension 3, [9] for dimension 4, and [6] for dimensions $\geq$5), we can find a homeomorphism τ_2 of $\mathbb{R}^n$ which agrees with τ_1 on a small ball neighborhood of z and is the identity outside a larger ball neighborhood of z. Hence, we can assume there is a neighborhood N' of z contained in N such that $\tau_2(A(z))\supset h_{i-1}^{-1}(\pi(E_i))\cap N'$ and $\tau_2=\mathrm{id}$ outside N. Hence, $h_{i-1}^{-1}(\pi(E_i))\cap N'\subset\{z\}\cup\tau_2(V'(z))$. Also $h_{i-1}^{-1}(\pi(E_i))\subset\{z\}\cup h_{i-1}^{-1}(P_i)$ and $\mathrm{cl}(h_{i-1}^{-1}(P_i))\cap h_{i-1}^{-1}(\pi(C_i))=\{z\}$. Thinking of the tame arc $h_{i-1}^{-1}(\pi(F_i))$ as a straight line segment, we see that there is a homeomorphism τ_3

of $\mathbb{R}^n$ which moves points away from $h_{i-1}^{-1}(\pi(C_i))$ toward $h_{i-1}^{-1}(\pi(E_i))$ and takes each round sphere centered at z onto itself such that

(12) there is a neighborhood N'' of z in N' such that $\tau_3(\mathrm{cl}(h_{i-1}^{-1}(P_i)) \cap N'') \subset \{z\} \cup \tau_2(V'(z))$,

(13) $\tau_3(z) = z$, and

(14) $\tau_3 = \mathrm{id}$ outside N.

Then $\tau_3^{-1}(z) = z$, $\tau_3^{-1} = \mathrm{id}$ outside N, and $\{z\} \cup \tau_3^{-1} \circ \tau_2(V'(z))$ contains $\mathrm{cl}(h_{i-1}^{-1}(P_i)) \cap N''$. Choose a neighborhood M_i of y_i such that $\mathrm{cl}(h_{i-1}^{-1}(M_i)) \subset N''$. Then $\tau_3^{-1} \circ \tau_2(V'(z))$ contains $h_{i-1}^{-1}(\mathrm{cl}(P_i \cap M_i) - \{y_i\})$. Define the homeomorphism h_i of $\mathbb{R}^n$ by $h_i = h_{i-1} \circ \tau_3^{-1} \circ \tau_2$. The properties of N together with the fact that $\tau_3^{-1} \circ \tau_2$ is supported on N imply that h_i satisfies conditions (8) and (9) and that $h_i(V'(h_j^{-1}(y_j)))$ contains $\mathrm{cl}(P_j \cap M_j) - \{y_j\}$ for $1 \leq j < i$. Also, τ_2 and τ_3 have obviously been chosen to insure that $h_i(V'(h_i^{-1}(y_i)))$ contains $\mathrm{cl}(P_i \cap M_i) - \{y_i\}$. So h_i satisfies condition (11). □

6. Questions

Let $\mathcal{S}$ be the collection of all smooth subsets of $\mathbb{R}^n$.

(1) What are the topological properties of $\mathbb{R}^n$ with the topology $\mathcal{T}_{\mathcal{S}}$? For instance, is it regular? normal? paracompact? first countable? second countable? separable? locally compact? connected? locally connected? contractible? locally contractible? What is its dimension?

The second question is a reformulation of questions (1) and (2) of [2].

(2) Suppose $f: \mathbb{R}^n \to \mathbb{R}^n$ is an injective function whose restriction to each smooth subset of $\mathbb{R}^n$ is continuous. Let $Z = \{x \in \mathbb{R}^n : f \text{ is discontinuous at } x\}$.

(a) Can Z be uncountable?

(b) Can Z be a (tame) Cantor set?

(c) Can $\dim Z > 0$?

Appendix

First we prove that if $1 \leq k < n$ and M is a C^2 regular k-manifold in $\mathbb{R}^n$, then M is a smooth set. Let $p \in M$. Then there is a C^2 diffeomorphism $\psi: U \to V$ from a neighborhood U of p in $\mathbb{R}^n$ to an open subset V of $\mathbb{R}^k \times \mathbb{R}^{n-k}$ such that $\psi(M \cap U) = (\mathbb{R}^k \times \{0\}) \cap V$ [4, p. 215]. Define $\pi: \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}$ by $\pi(x_1, \ldots, x_k, y_1, \ldots, y_{n-k}) = y_1$. Then $\pi \circ \psi: U \to \mathbb{R}$ is a C^2 map such that $M \cap U \subset (\pi \circ \psi)^{-1}(0)$. Also, for each $q \in U$, since $(\pi \circ \psi)'(q) = \pi'(\psi(q)) \cdot \psi'(q)$, $\psi'(q)$ is rank n and $\pi'(\psi(q))$ is rank 1, then $(\pi \circ \psi)'(q)$ is rank 1; so $\pi \circ \psi$ has a nonzero first order partial derivative at q.

Second, we indicate how the version of Taylor's formula given in Section 2 is derived from the following version which is commonly found in advanced calculus texts. Let $r \geq 1$, let U be an open subset of $\mathbb{R}^n$, let $f: U \to \mathbb{R}$ be a C^{r+1} map, and let

$p \in U$. If $x \in \mathbb{R}^n$ such that U contains the straight line segment from p to $p+x$, then there is a $\theta \in (0,1)$ such that

$$f(p+x) = \sum_{k=0}^{r} \frac{1}{k!} D^k f(p)(x, \ldots, x) + \frac{1}{(r+1)!} D^{r+1} f(p+\theta x)(x, \ldots, x)$$

[7, p. 179]. Here $D^k f(p) : (\mathbb{R}^n)^k \to \mathbb{R}$ is the symmetric multilinear function

$$D^k f(p)(v^1, \ldots, v^k) = \sum_{\substack{1 \leq i_j \leq n \\ 1 \leq j \leq k}} \frac{\partial^k f}{\partial x_{i_1}, \ldots, \partial x_{i_k}} (p) v^1_{i_1} \ldots v^k_{i_k}$$

where $v^j = (v^j_1, \ldots, v^j_n) \in \mathbb{R}^n$ for $1 \leq j \leq k$. Since $D^k f(p)$ is symmetric, then by combining like terms, $D^k(p)(x, \ldots, x)$ can be simplified to

$$\sum_{\substack{a \in \omega^n \\ \|a\| = k}} \frac{k!}{a!} f^{(a)}(p) x^a.$$

Substituting this expression for $D^k f(p)(x, \ldots, x)$ in the above version of Taylor's formula yields the version of Taylor's formula given in Section 2.

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